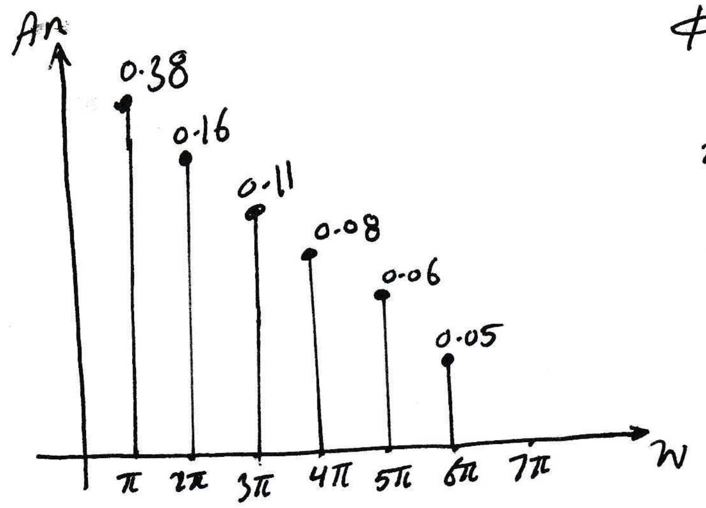
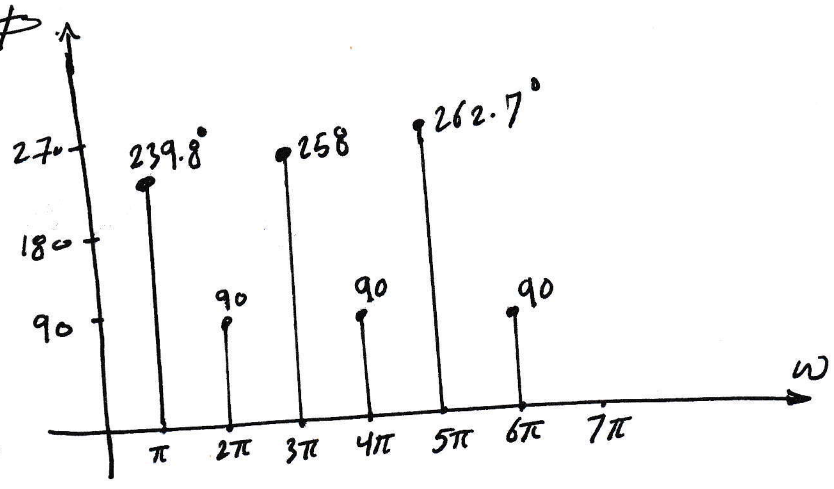




Symmetry Considerations :-

1- Even Symmetry :-

A function $f(t)$ is even if its plot is Symmetrical about the vertical axis.

$$f(t) = f(-t)$$

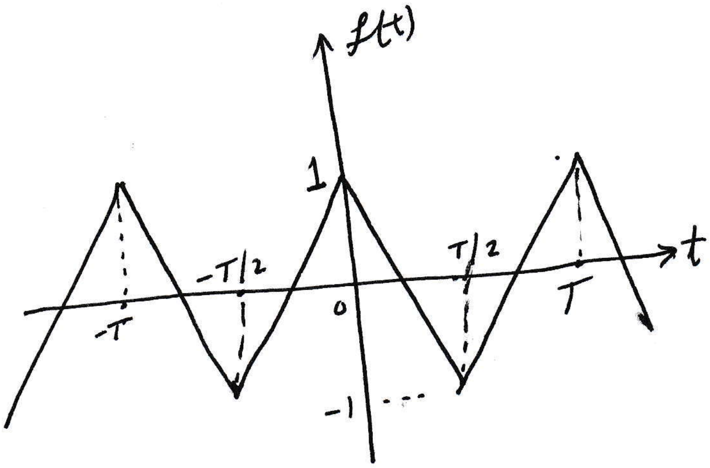
Example of even function are $t^2, t^4, t^6, \ldots,$ and $\cos t$.

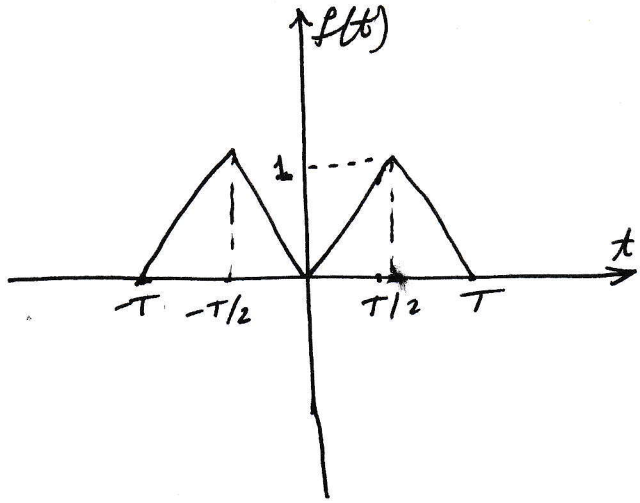
$$\int_{-T/2}^{T/2} f(t)\, dt = 2 \cdot \int_0^{T/2} f(t)\, dt.$$

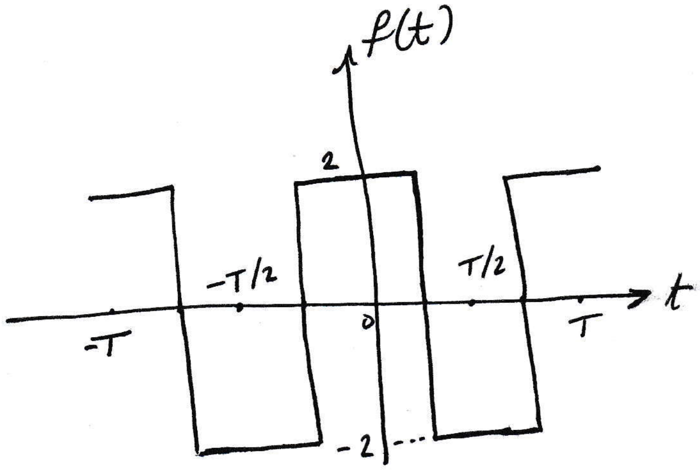
$$a_0 = \frac{2}{T_0} \int_0^{T/2} f(t)\, dt$$

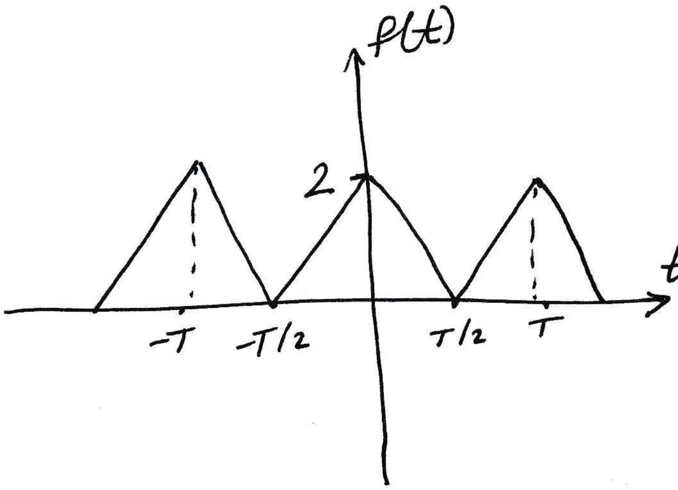
$$a_n = \frac{4}{T} \int_0^{T/2} f(t) \cos n w_0 t\, dt$$

$$b_n = 0$$









The Fourier Series is called "a Fourier Cosine Series" at this Case.

2- Odd Symmetry

A function $f(t)$ is said to be odd if its plot is symmetrical about ~~vertical~~ horizontal axis:-

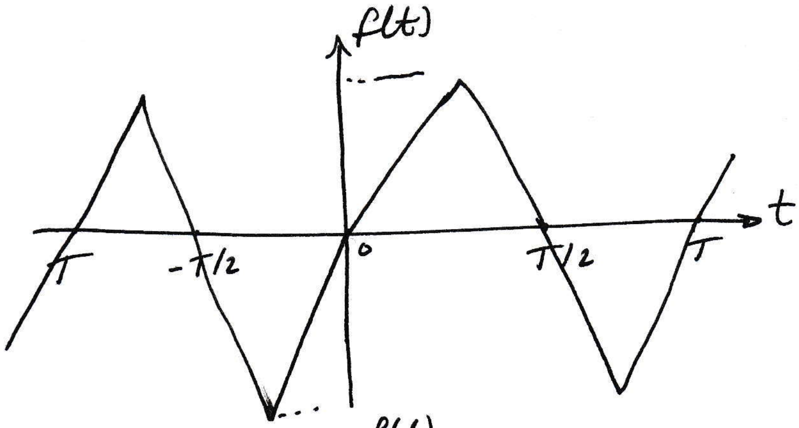
$$f(-t) = -f(t)$$

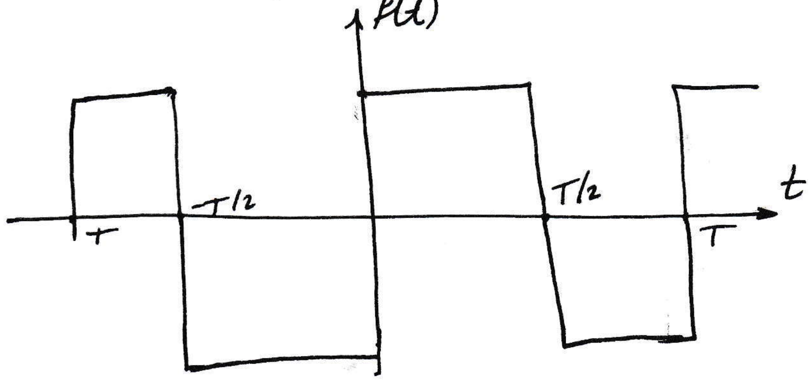
Examples of odd function are $t, t^3, t^5, \ldots,$ and $\sin t$

$$\int_{-T/2}^{T/2} f(t)\,dt = 0$$

$$a_0 = 0, \quad a_n = 0$$

$$b_n = \frac{4}{T}\int_0^{T/2} f(t)\,\sin n\omega_0 t\,dt$$





if the question is to find a_0, a_n, and b_n, you must find all of them even you know the function is odd or even.

$$f(t) = \underbrace{a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t}_{\text{even}} + \underbrace{\sum_{n=1}^{\infty} b_n \sin n\omega_0 t}_{\text{odd}}$$

Properties of Odd and Even function :-

1- the product of even with even, the function o/p is even.

2- " " " odd " odd, " " " " odd.

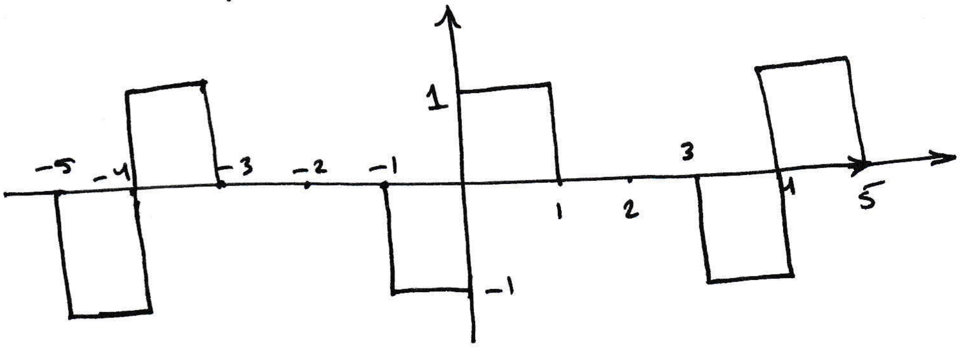
3- " " " even " odd, " " " " odd.

4- The sum or difference of two odd functions is an odd function.

5- the sum or difference of two even functions is an even function.

6- the sum (or difference) of even function and an odd function is neither even nor odd.

EX Find the Fourier Series expansion of $f(t)$ given in Figure below.



$a_0 = 0$, $a_n = 0$

$$\therefore b_n = \frac{4}{T} \int_0^{T/2} f(t) \sin n\omega_o t \, dt$$

$$= \frac{4}{T} \left[\int_0^1 1 \cdot \sin \frac{n\pi}{2} t \, dt + \int_1^2 0 \cdot \sin \frac{n\pi}{2} t \, dt \right]$$

$$= -\frac{2}{n\pi} \cos \frac{n\pi}{2} \Big|_0^1 = \frac{2}{n\pi} \left(1 - \cos \frac{n\pi}{2}\right)$$

$$\therefore f(t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left(1 - \cos \frac{n\pi}{2}\right) \sin \frac{n\pi}{2} t$$

EX A voltage Source has a periodic wave form defined over its period as $v(t) = t(2\pi - t)$ volt for $0 < t < 2\pi$.

Find the Fourier Series for this voltage?

$$v(t) = 2\pi t - t^2 \qquad 0 < t < 2\pi$$

$T = 2\pi$

$$a_0 = \frac{1}{T} \int_0^T f(t) dt = \frac{1}{2\pi} \int_0^{2\pi} (2\pi t - t^2) dt$$

$$= \frac{1}{2\pi} \left(\pi t^2 - \frac{t^3}{3}\right) \Big|_0^{2\pi} \Rightarrow \frac{4\pi^2}{2\pi} \left(1 - \frac{2}{3}\right) \Rightarrow \boxed{a_0 = \frac{2\pi^2}{3}}$$

$$a_n = \frac{2}{T}\int_0^T (2\pi t - t^2)\cos nt\,dt$$

$$= \frac{1}{\pi}\left[\frac{2\pi}{n^2}\cos(nt) + \frac{2\pi t}{n}\sin(nt)\right]\Big|_0^{2\pi}$$

$$a_n = \frac{-1}{\pi n^3}\left\{2nt\cos nt - 2\sin(nt) + n^2t^2\sin(nt)\right\}\Big|_0^{2\pi}$$

$$a_n = \frac{2}{n^2}(1-1) - \frac{1}{\pi n^3}\left[4n\pi\cos(2\pi n)\right] = \frac{-4}{n^2}$$

$$b_n = \frac{2}{T}\int_0^T (2\pi t - t^2)\sin n\omega_0 t = \frac{1}{\pi}\int_0^{2\pi}(2\pi t - t^2)\sin(nt)\,dt$$

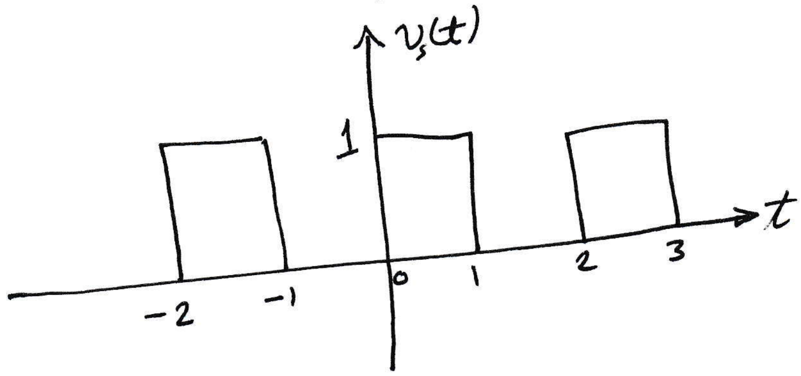
$$b_n = \frac{2n}{\pi}\frac{1}{n^2}\left[\sin(nt) - nt\cos(nt)\right]\Big|_0^{\pi} - \frac{1}{n^3\pi}\left[2\pi t\sin(nt)dt\right]$$

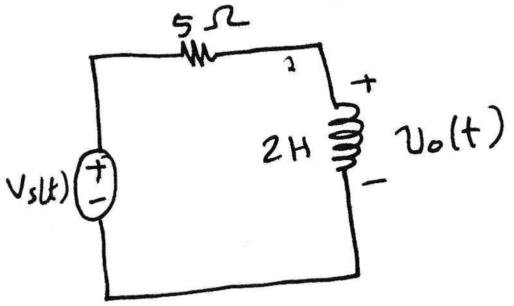
$$+ 2\cos(nt) - n^2t^2\cos(nt)\Big]\Big|_0^{2\pi}$$

$$b_n = \frac{-4\pi}{n} + \frac{4\pi}{n} = 0$$

Circuit application :-

EX Find the response $v_o(t)$ of the circuit below.





$$v_s(t) = \frac{1}{2} + \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{1}{n} \sin n\pi t \quad , n = 2k-1$$

even $n = 2k$
odd $n = 2k-1$

$w_n = n w_o = n\pi \quad \therefore w_o = \pi$

and by voltage division

$$v_o = \frac{j w_n L}{R + j w_n L} V_s = \frac{j 2n\pi}{5 + j 2n\pi} V_s$$

for the DC Component

$$V_s = \frac{1}{2} \Rightarrow V_o = 0$$

at D.C. the inductance be short circuit as wire.

For the harmonic :-

$$V_s = \frac{2}{n\pi} \angle -90$$

$$V_o = \frac{2n\pi \angle 90}{\sqrt{25 + 4n^2\pi^2} \angle \tan^{-1}\frac{2n\pi}{5}} * \frac{2}{n\pi} \angle -90$$

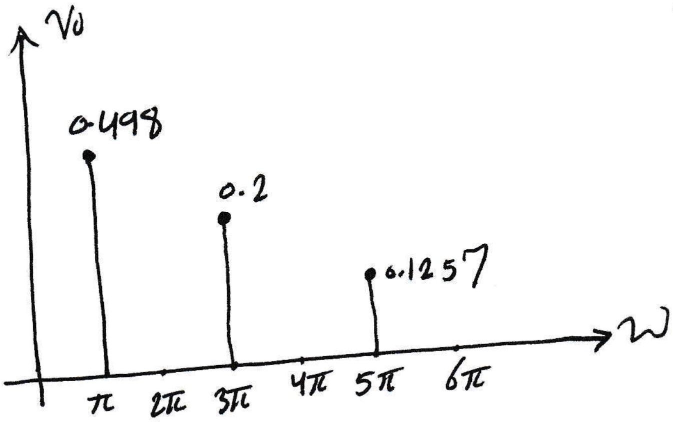
$$= \frac{4 \angle -\tan^{-1} 2n\pi/5}{\sqrt{25 + 4n^2\pi^2}}$$

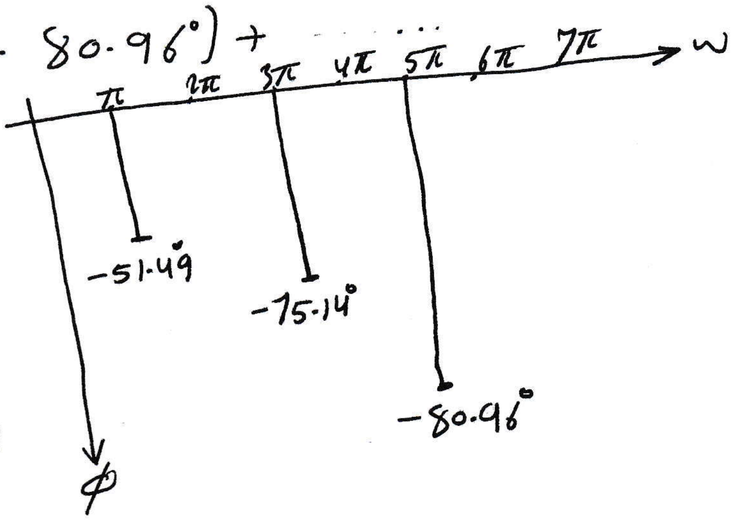
In the time domain

$$V_o(t) = \sum_{K=1}^{\infty} \frac{4}{\sqrt{25+4n^2\pi^2}} \cos\left(n\pi t - \tan^{-1}\frac{2n\pi}{5}\right)$$

for $K = 1, 2, 3$ or $n = 1, 3, 5$ of the harmonic (odd)

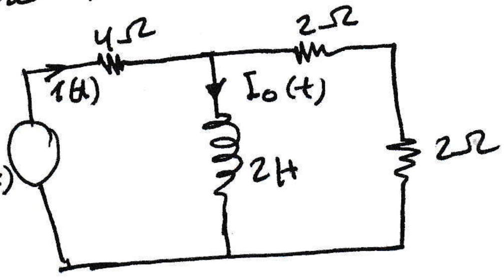
$$V_o(t) = 0.4981 \cos(\pi t - 51.49^\circ) + 0.2051 \cos(5\pi t - 75.14^\circ)$$
$$+ 0.1257 \cos(5\pi t - 80.96^\circ) + \cdots$$





EX Find the response $i_o(t)$ in the circuit in figure below if the input voltage $V(t)$ has the Fourier Series.

$$V_s(t) = 1 + \sum_{n=1}^{\infty} \frac{2(-1)^n}{1+n^2}(\cos nt - n \sin nt)$$



$$V_s(t) = 1 + \sum_{n=1}^{\infty} \frac{2(-1)^n}{\sqrt{1+n^2}} \cos(nt + \tan^{-1} n)$$

$$= 1 - 1.414 \cos(t + 45^\circ) + 0.8944 \cos(2t + 63.45^\circ).$$
$$- 0.6345 \cos(3t + 71.56) - 0.481 \cos(4t + 78.7^\circ) + \cdots$$

$$w = 1, \Rightarrow Z = 4 + (jwn2 \,//\, 4)$$

$$Z = 4 + \frac{j\omega n 8}{4 + j\omega n 2} = \frac{8 + j\omega n 8}{2 + j\omega n}$$

Input Current

$$I = \frac{v}{Z} = \left(\frac{2 + j\omega n}{8 + j\omega n 8}\right) v$$

By Current division

$$I_o = \frac{4}{4 + j\omega n 2} I = \frac{V}{4 + j\omega n 4}$$

$$I_o = \frac{v}{4\sqrt{1+n^2} \angle \tan^{-1} n}$$

for dc Component:-

$$V = 1 \Rightarrow I_o = \frac{v}{4} = \frac{1}{4}$$

for the nth harmonic

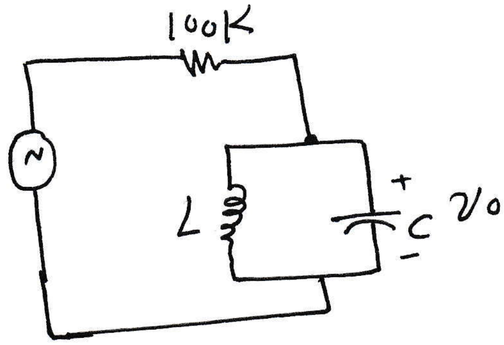
$$v = \frac{2(-1)^n}{\sqrt{1+n^2}} \angle \tan^{-1} n$$

$$I_o = \frac{1}{4\sqrt{1+n^2} \angle \tan^{-1} n} * \frac{2(-1)^n}{\sqrt{1+n^2}} \angle \tan^{-1} n = \frac{(-1)^n}{2(1+n^2)}$$

the time domain

$$i_o(t) = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n}{2(1+n^2)} \cos n t \quad \text{Amp.}$$

EX For figure below.



$$V_s(t) = \begin{cases} 10 & 0 < t < \pi \text{ msec} \\ 0 & \pi_{msec} < t < 2\pi \text{ msec} \end{cases}$$

$L = 1\,H, \quad C = 1\,\mu f$

Determine the value of $V_o(t)$.

$$\omega_o = \frac{2\pi}{T} \Rightarrow \frac{2\pi}{2\pi} = 1$$

$$V_o(t) = V_s(t) * \frac{1/j\omega n C}{100 + \frac{1}{j\omega n C}}$$

$$V_o(t) = \frac{V_s(t)}{1 + j0.5n}$$

for D.C $V_o(t) = 10$

for harmonic

$\omega = 1$

$$V_o(t) = \frac{10}{\sqrt{1 + (0.5n)^2}\ \angle \tan^{-1} 0.5n} = 7 \angle 45$$

$\omega = 3$

$$V_o(t) = \frac{10}{\sqrt{1 + (0.5 \times 3)^2}\ \angle \tan^{-1} 0.5 \times 3} = 5.5 \angle 56.3^\circ$$

$\therefore$ $V_o(t)$ in the time domain

$$V_o(t) = 10 + 8.9\cos(nt + 26.5^\circ) + 7\cos(nt + 45^\circ) + 5.5\cos(nt + 56.3^\circ)$$

OR $$V_o(t) = 10 + \sum \frac{10}{\sqrt{1 + (0.5n)^2}} \cos(nt + \tan^{-1} 0.5n)$$

Average power and RMS values :-

The voltage and current in amplitude-phase form :-

$$v(t) = V_{d.c} + \sum_{n=1}^{\infty} V_n \cos(n w_0 t - \phi_n) \quad (1)$$

$$I(t) = I_{d.c} + \sum_{m=1}^{\infty} I_n \cos(m w_0 t - \phi_m) \quad (2)$$

the average power $P = \frac{1}{T}\int_0^T v i \, dt \quad (3)$

$$\boxed{P = V_{d.c} * I_{d.c} + \frac{1}{2}\sum_{n=1}^{\infty} V_n I_n \cos(\phi_n - \phi_m)}$$

the R.M.S value or (effective value is)

$$F_{r.m.s} = \sqrt{\frac{1}{T}\int_0^T f^2(t)\, dt}$$

$$F_{r.m.s}^2 = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} A_n^2$$

$$F_{r.m.s} = \sqrt{a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} A_n^2}$$

$$\boxed{F_{r.m.s} = \sqrt{a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} (a_n^2 + b_n^2)}} \rightarrow \text{Parsevals theorem}$$

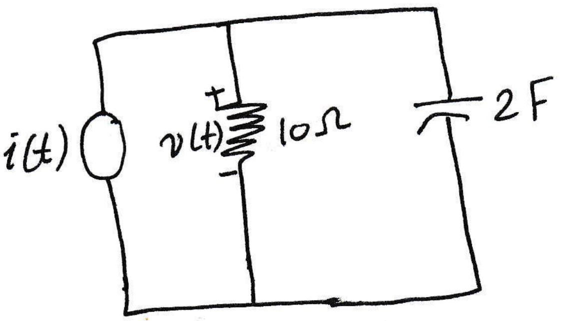
If the $f(t)$ is the current through a resistance R, then the power dissipated in the resistor is :-

$$\boxed{P = R\, F_{r.m.s}^2}$$

or if $f(t)$ is the voltage across a resistor R, the Power dissipated in the resistor is :-

$$P = \frac{F_{r.m.s}^2}{R}$$

EX Determine the average power Supplied to the current in figure below if $i(t) = 2 + 10\cos(t+10) + 6\cos(3t+35)$ A and then find the estimate for the R.M.S value for the voltage?



Input impedance is

$$Z = 10 \,//\, \frac{1}{j2w} = \frac{10\left(\frac{1}{j2w}\right)}{10 + \frac{1}{j2w}}$$

$$\therefore Z = \frac{10}{1 + j20w}$$

$$V = IZ = \frac{10\, I}{\sqrt{1+400w^2}\ \angle \tan^{-1} 20w}$$

For the d.c. Component, $I = 2A \Rightarrow V = 10 * 2 = 20V$

For $w = 1$

$$I = 10\angle 10^\circ \Rightarrow V = \frac{10(10\angle 10)}{\sqrt{1+400}\ \angle \tan^{-1} 20} = 5\angle -77.14^\circ$$

for $w = 3$

$$I = 6\angle 45^\circ \Rightarrow V = \frac{10(6\angle 45}{\sqrt{1+3600}\ \angle \tan^{-1} 60} = 1\angle -44.05^\circ$$

In the time domain

$$v(t) = 20 + 5\cos(t - 77.14^\circ) + 1\cos(3t - 44.05)$$

$$P = V_{dc} * I_{dc} + \frac{1}{2}\sum_{n=1}^{\infty} V_n I_n \cos(\theta_n - \phi_n)$$

$$\therefore P = 20 * 2 + \frac{1}{2} * 5 * 10 \cos(-77.14 - 10) + \frac{1}{2} * 1 * 6 \cos(-44.05 - 35)$$

$$P = 40 + 1.247 + 6.5 = 41.5 \text{ w}$$

$$\text{or } P = \frac{(F_{r.m.s})^2}{10} = 41.3 \Rightarrow \frac{(20.322)^2}{10} = 41.3$$

$$V_{r.m.s} = \sqrt{a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} A_n^2}$$

$$V_{r.m.s} = \sqrt{a^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)}$$

$$= \sqrt{(20)^2 + \frac{1}{2}[5^2 + 1]}$$

$$= 20.322 \text{ V}$$

Exponential Fourier Series :-

The Sine and Cosine function Can be represented in the exponential form using Euler's adentity.

$$\cos n\omega_0 t = \frac{1}{2}\left[e^{jn\omega_0 t} + e^{-jn\omega_0 t}\right] \quad \text{—— (1)}$$

$$\sin n\omega_0 t = \frac{1}{j2}\left[e^{jn\omega_0 t} - e^{-jn\omega_0 t}\right] \quad \text{—— (2)}$$

We have $f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$ .. (3)

Sub (1) + (2) in (3)

$$f(t) = a_0 + \frac{1}{2}\sum_{n=1}^{\infty}[a_n - jb_n]\,e^{jn\omega_0 t} + [a_n + jb_n]\,e^{-j\omega_0 nt}$$

if we define a new coefficient C_n so that

$$C_0 = a_0, \quad C_n = \frac{a_n - jb_n}{2}, \quad C_{-n} = C_n^{*} = \frac{a_n + jb_n}{2}$$

then $f(t)$ becomes

$$f(t) = C_0 + \sum_{n=1}^{\infty}\left(C_n e^{jn\omega_0 t} + C_{-n} e^{-jn\omega_0 t}\right)$$

$$\therefore \boxed{f(t) = \sum_{n=-\infty}^{\infty} C_n \cdot e^{jn\omega_0 t}}$$

this is Complex or exponential Fourier Series

$$\therefore \boxed{C_n = \frac{1}{T}\int_0^T f(t)\cdot e^{-jn\omega_0 t}\,dt}$$

The plot of the magnitude and phase of "C_n" versus $n\omega_0$ are called the "Complex Amplitude Spectrum" and "complex phase spectrum" of $f(t)$.

The two spectrum form the "Complex frequency spectrum" of $f(t)$.

$$A_n \angle \phi_n = a_n - j b_n = 2 C_n$$

$$C_n = |C_n| \angle \phi_n = \frac{\sqrt{a_n^2 + b_n^2}}{2} \angle -\tan^{-1}\frac{b_n}{a_n}$$

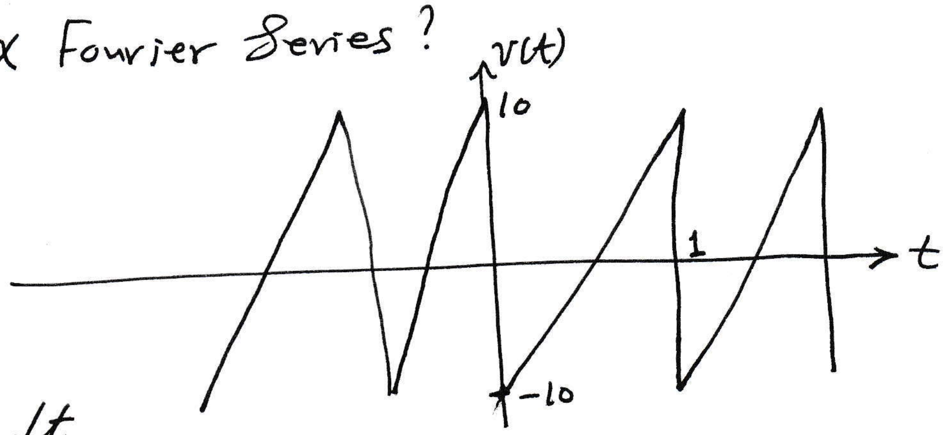
R.M.S value of a periodic signal $f(t)$ as $F_{r.m.s}^2$

$$F_{r.m.s}^2 = \frac{1}{T}\int_0^T f^2(t)\,dt = \frac{1}{T}\int_0^T f(t)\left[\sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}\right]dt$$

$$F_{r.m.s}^2 = \sum_{n=-\infty}^{\infty} C_n \left[\frac{1}{T}\int_0^T f(t)\, e^{jn\omega_0 t}\right]$$

$$F_{r.m.s}^2 = \sum_{n=-\infty}^{\infty} |C_n|^2$$

EX Given the Sawtooth voltage wave shape shown below, Find its Complex Fourier Series?



$$C_n = \frac{1}{T}\int_0^T v(t)\cdot e^{-jnw_0 t}\,dt$$

$T = 1$ and $v(t) = 20t - 10 \quad 0 < t < 1$

$w = 2\pi$

$$C_n = \frac{1}{1}\int_0^1 (20t-10)\, e^{-j2n\pi t}\,dt$$

$$= 20\int_0^1 t\, e^{-j2n\pi t}\,dt - 10\int_0^1 e^{-j2\pi n t}\,dt$$

$$= 20\left[\frac{t e^{-j2\pi n t}}{-j2\pi n} - \frac{e^{-j2\pi n t}}{(-j2\pi n)^2}\right]\Bigg|_0^1 - 10\,\frac{e^{-j2\pi n t}}{-j2\pi n}\Bigg|_0^1$$

$$= 20\left[\frac{e^{-j2n\pi}}{-j2\pi n} + \frac{e^{j2\pi n}}{4\pi^2 n^2} - \frac{1}{4\pi^2 n^2}\right] - 10\left[\left(e^{-j2\pi n}\,\frac{j}{2\pi n}\right) - \frac{j}{2\pi n}\right]$$

$$= 20\left[\frac{j}{2\pi n} + \frac{1}{4\pi^2 n^2} - \frac{1}{4\pi^2 n^2}\right] - 10\left[\frac{j}{2\pi n} - \frac{j}{2\pi n}\right]$$

$$= j\,\frac{20}{2\pi n}$$

$$C_0 = \frac{1}{1}\int_0^1 (20t-10)\, e^{0t}\,dt \Rightarrow \boxed{C_0 = 0}$$

$$\therefore\ v(t) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{j10}{n\pi}\, e^{j2\pi n t}$$

EX determine the exponential Fourier Series for

$f(t) = t^2, \quad -\pi < t < \pi$?

$\omega_0 = \frac{2\pi}{T} = 1$

$$C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} t^2 \cdot e^{-jnt}\, dt$$

$\vdots$

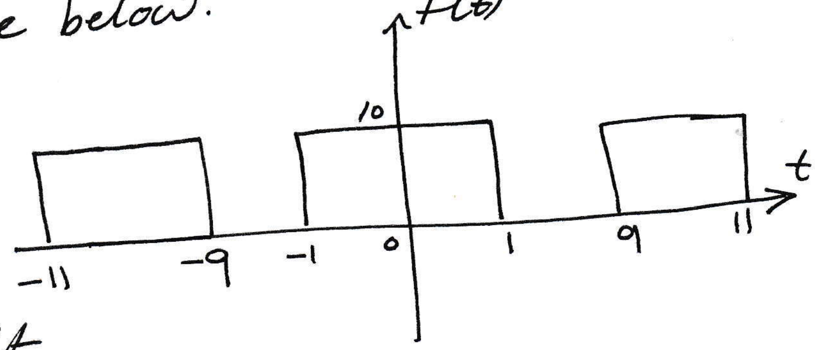
$C_n = 2\cos(n\pi)/n^2 = 2*(-1^n/n^2) \quad n \neq 0$

For $n = 0$

$$C_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} t^2\, dt = \frac{\pi^2}{3}$$

$$\therefore f(t) = \frac{\pi^2}{3} + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{2*(-1)^n}{n^2} e^{jnt}$$

EX Find the exponential Fourier Series Complex frequency Spectrum for figure below.



$$C_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t)\, e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{10} \int_{-1}^{1} 10\, e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{-jn\omega_0} e^{-jn\omega_0 t}\Big|_{-1}^{1} = \frac{1}{-jn\omega_0}\left(e^{-jn\omega_0} - e^{jn\omega_0}\right)$$

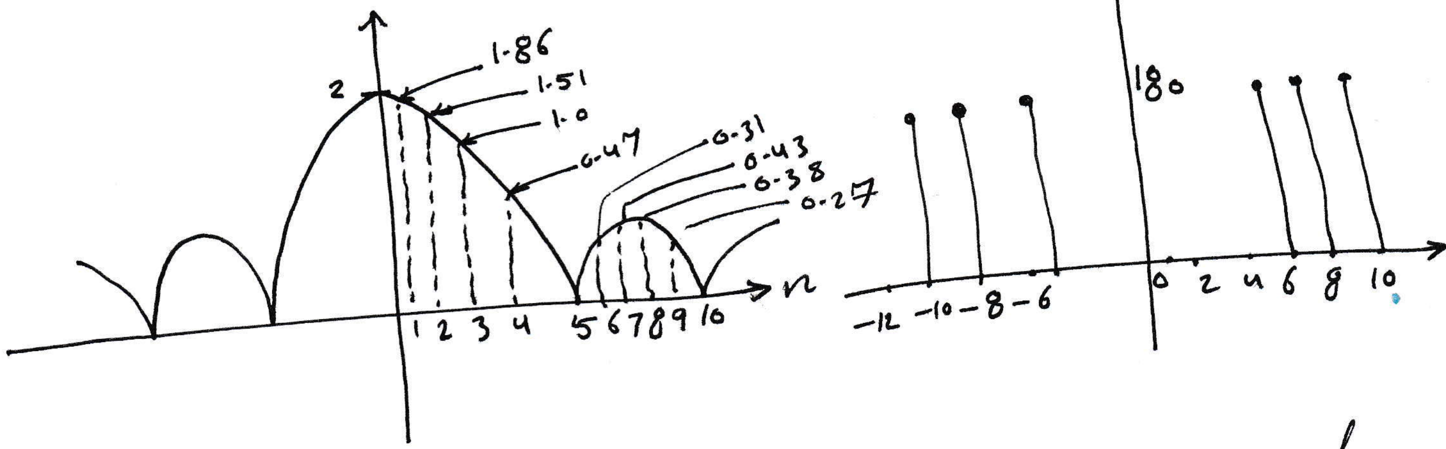
$$= \frac{2}{n w_0} \cdot \frac{e^{jnw_0} - e^{-jnw_0}}{2j} = \frac{2}{n w_0} \sin n w_0 \quad , \quad w_0 = \frac{\pi}{5}$$

$$\therefore \; C_n = 2 \, \frac{\sin n\pi/5}{n\pi/5}$$

$$|C_n| = 2 \left| \frac{\sin n\pi/5}{n\pi/5} \right| \text{ amplitude}$$

$$\phi_n = \begin{cases} 0 & \frac{\sin n\pi/5}{n\pi/5} > 0 \\ 180 & \frac{\sin n\pi/5}{n\pi/5} < 0 \end{cases}$$

by L'Hopitals rule.



EX Find the exponential Fourier Series expansion of the period function $f(t) = e^t$, $0 < t < 2\pi$ and then find the Complex frequency spectrum.

$T = 2\pi$, $w_0 = 1$

$$C_n = \frac{1}{T} \int_0^T f(t) \cdot e^{-jnw_0 t} \, dt = \frac{1}{2\pi} \int_0^{2\pi} e^t e^{-jnt} \, dt$$

$$= \frac{1}{2\pi} \cdot \frac{1}{1-jn} \, e^{(1-jn)t} \Big|_0^{2\pi} = \frac{1}{2\pi(1-jn)} \left[e^{2\pi} e^{-j2\pi n} - 1 \right]$$

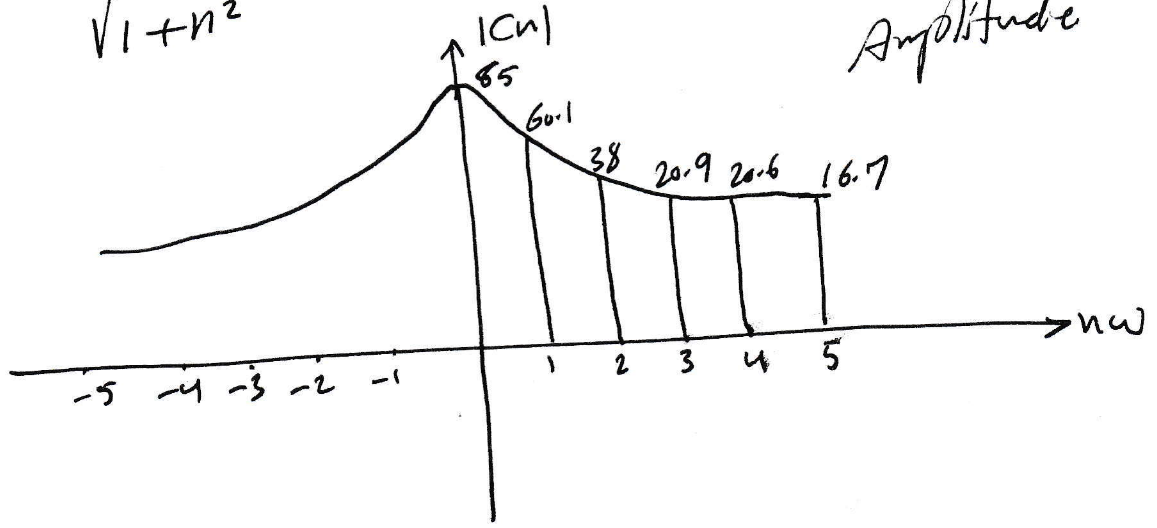
by Eulers identity $\Rightarrow e^{-j2\pi n} = \cos 2\pi n - j\sin 2\pi n = 1 - j0 = 1$

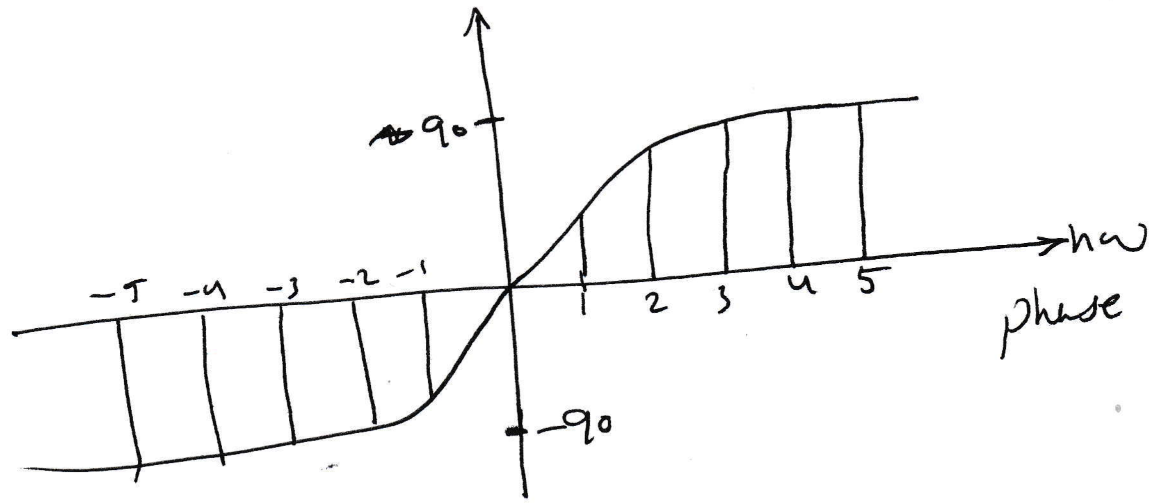
$$\therefore C_n = \frac{1}{2\pi(1-jn)}\left[e^{2\pi} - 1\right] = \frac{85}{1-jn}$$

the Complex fourier Series is :-

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{85}{1-jn} e^{jnt}, \quad C_n = |C_n| \angle \phi$$

$$|C_n| = \frac{85}{\sqrt{1+n^2}}, \quad \phi = \tan^{-1} n$$





Fourier Integral

Fourier Series are powerful tools for problems involving functions that are "periodic" or are of interest on a "finite interval only". Many problems involve functions that are "nonperiodic" and are of interest on the whole x-axis, to solve this problem, the idea is to extend the method of Fourier Series to such functions such as "Fourier integrals".

EX Rectangular Wave

Consider the periodic rectangular wave $f_L(x)$ of period $2L > 2$ given by

$$f_L(x) = \begin{cases} 0 & \text{if } -L < x < -1 \\ 1 & \text{if } -1 < x < 1 \\ 0 & \text{if } 1 < x < L \end{cases}$$

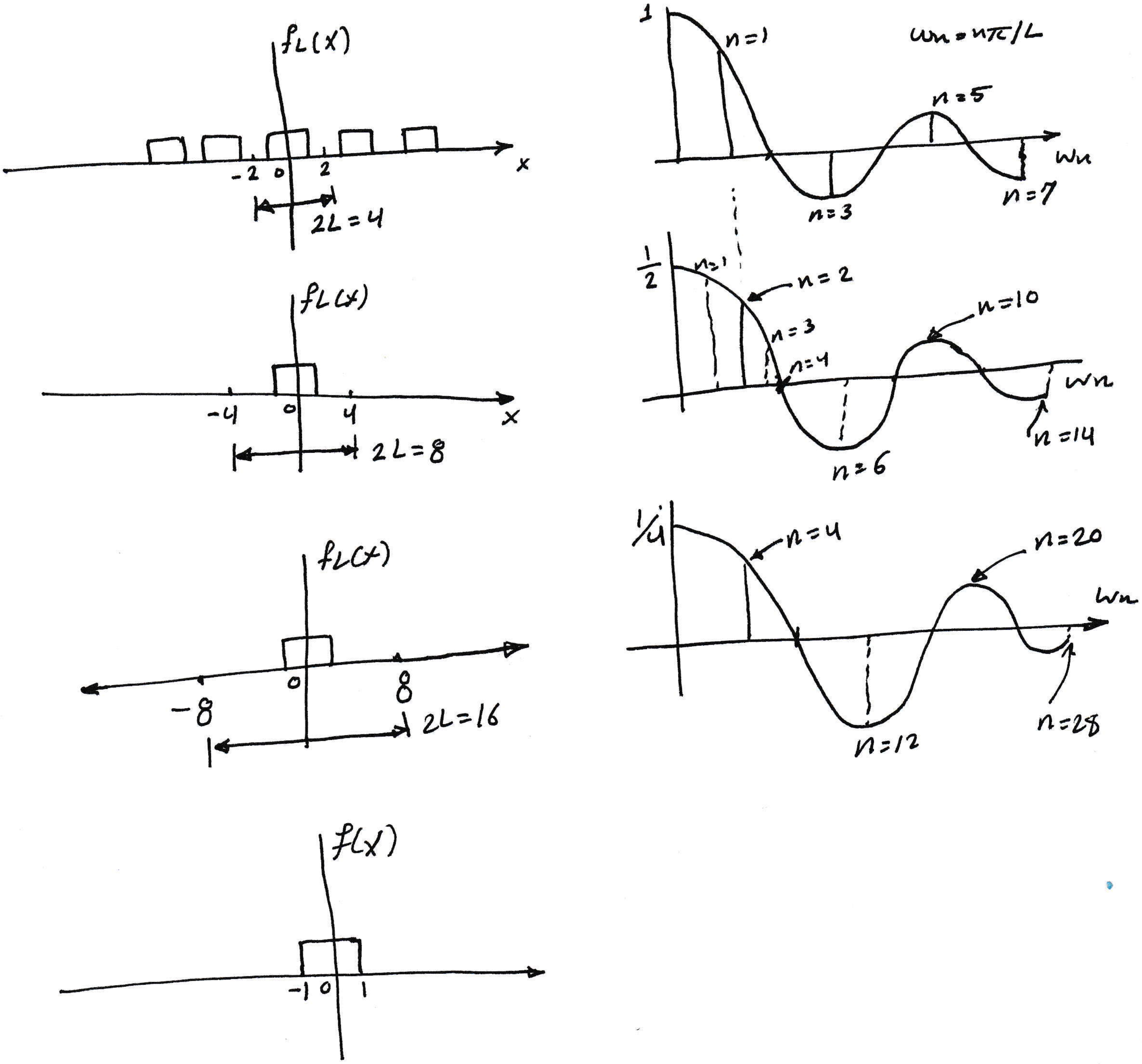
or
$$f(x) = \lim_{L\to\infty} f_L(x) = \begin{cases} 1 & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

the nonperiodic function $f(x)$ $2L = 4, 8, 16$, which we obtain from f_L if we let $L \to \infty$,

We now explore what happens to the Fourier coefficients of f_L as L increases, Since f_L is even, $b_n = 0$ for all n.

$$a_0 = \frac{1}{2\pi} \int_{-1}^{1} dx = \frac{1}{L}$$

$$a_n = \frac{1}{L} \int_{-1}^{1} \cos \frac{n\pi x}{L}\, dx = \frac{2}{L} \int_{0}^{1} \cos \frac{n\pi x}{L}\, dx = \frac{2}{L} \frac{\sin(n\pi/L)}{n\pi/L}$$



fL(x)
-2
0
2
x
2L=4
fL(x)
-4
0
4
x
2L=8
fL(x)
-8
0
8
2L=16
f(x)
-1
0
1
1
n=1
wn = nπ/L
n=5
wn
n=3
n=7
1/2
n=1
n=2
n=10
n=3
n=4
wn
n=6
n=14
1/4
n=4
n=20
wn
n=12
n=28